

YOSHIKAWA MOVES ON MARKED GRAPHS VIA ROSEMAN'S THEOREM

OLEG CHTERENTAL

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ABSTRACT. Yoshikawa [Yo] conjectured that a certain set of moves on marked graph diagrams generates the isotopy relation for surface links in \mathbb{R}^4 , and this was proved by Swenton [S] and Kearton and Kurlin [KK]. In this paper, we find another proof of this fact for the case of 2-links (surface links with spherical components). The proof involves a construction of marked graphs from branch-free broken surface diagrams, and a version of Roseman's theorem [R] for branch-free broken surface diagrams of 2-links. As an application, we find that Brunnian 2-links are ribbon.

1. INTRODUCTION

For a smooth oriented manifold M we denote by $\text{Diff}^+(M)$ the group of orientation-preserving diffeomorphisms of M and by $\text{Diff}_0^+(M)$ the path-component of the identity in $\text{Diff}^+(M)$. We orient \mathbb{R}^4 in the standard way. A **surface link** $L \subset \mathbb{R}^4$ is a smooth submanifold diffeomorphic to a closed surface (i.e. compact with no boundary). We say two surface links L_1 and L_2 are **isotopic** if there is $f \in \text{Diff}_0^+(\mathbb{R}^4)$ with $f(L_1) = L_2$. A **2-link** is a surface link where each component is a 2-sphere. Let \mathcal{L} be the set of surface links, \mathcal{L}^+ the set of surface links with orientable components, and \mathcal{L}_0 the set of 2-links.

In Section 2 we review the definitions of generic projections, broken surface diagrams, Roseman moves, and Roseman's theorem. We prove Theorem 2.9, a version of Roseman's theorem for branch-free broken surface diagrams of 2-links, stating that two isotopic branch-free broken surface diagrams of a 2-link are related by a finite sequence of *Ro1*, *Ro2*, *Ro5**, *Ro7*, *Br1* and *Br2* moves (see Figures 3 and 7). This theorem complements the results of Takase and Tanaka [TT], who find examples of isotopic branch-free broken surface diagrams of a 2-link that are not related by *Ro1*, *Ro2*, *Ro5** and *Ro7* moves alone.

In Section 3 we review marked graphs in \mathbb{R}^3 , marked graph diagrams in \mathbb{R}^2 , Yoshikawa moves on marked graph diagrams, *ab*-surfaces obtained from marked graphs, and prove some facts about marked

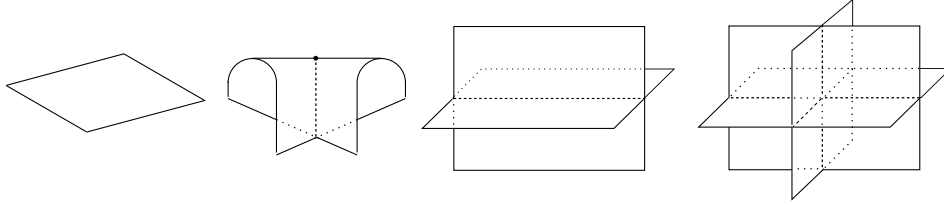


FIGURE 1. Non-singular, branch, double and triple points in a generic projection.

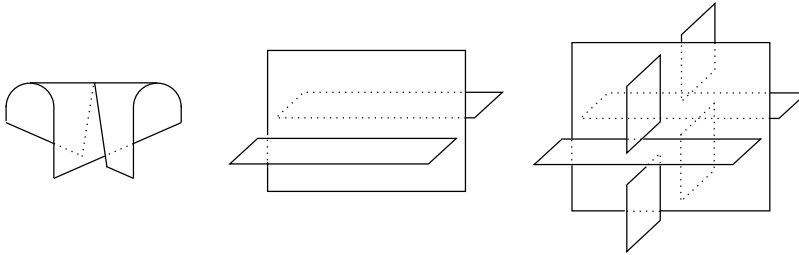


FIGURE 2. Broken surface diagrams of branch, double and triple points.

graphs and ab -surfaces. In Section 4 we describe a relationship between branch-free broken surface diagrams and ab -surfaces, and provide another proof that two marked graph diagrams describe isotopic 2-links if and only if they are related by a finite sequence of Yoshikawa moves (see Theorem 4.4). In Section 5 we prove Theorem 5.2 showing that Brunnian 2-links are ribbon.

ACKNOWLEDGEMENTS

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2. BROKEN SURFACE DIAGRAMS AND ROSEMAN'S THEOREM

Definition 2.1 reviews generic projections, broken surface diagrams and Roseman moves. Roseman's theorem is stated in Theorem 2.2. We use Lemma 2.7 to prove Theorem 2.9, a branch-free version of Roseman's theorem.

Definition 2.1. References for the definitions we present here can be found for example in Carter, Kamada and Saito [CKS], Kamada [Kam] and Roseman [R]. Let $\pi : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be the projection $(x, y, z, u) \mapsto$

(x, y, z) . If $L_1 \in \mathcal{L}$ there is $L_2 \in \mathcal{L}$ isotopic to L_1 such that a neighbourhood in \mathbb{R}^3 of any point of $\pi(L_2)$ is one of the four possibilities in Figure 1. Such a projection $\pi(L_2)$ will be called **generic**.

Assume $L \in \mathcal{L}$ is such that $D = \pi(L)$ is generic. Let $\text{sing}(D) \subset \mathbb{R}^3$ be the set of branch, double and triple points. If $p \in \text{sing}(D)$ is a double point then there exist $x_1 \neq x_2 \in L$ with $\pi(x_1) = \pi(x_2) = p$ and these two points are ordered by the u -coordinates of x_1 and x_2 . Similarly if $p \in \text{sing}(D)$ is a triple point then there exist $x_1 \neq x_2 \neq x_3 \in L$ with $\pi(x_1) = \pi(x_2) = \pi(x_3) = p$ and these three points are ordered by the u -coordinates of x_1, x_2 and x_3 . The orderings extend by continuity to neighbourhoods of the points in S . The projection D along with all of the above crossing information is called a **broken surface diagram**. Following Takase and Tanaka [TT], we refer to a broken surface diagram with no branch points as **branch-free**. We consider two broken surface diagrams **equivalent** if they differ by the action of $\text{Diff}_0^+(\mathbb{R}^3)$ and agree on crossing information. We may indicate the crossing information near a double-point by removing a neighbourhood of one of the pre-images with the convention that missing segments have greater u -coordinates as in Figure 2. Figure 3 depicts the Roseman moves on broken surface diagrams. Appropriate crossing information should be added for completeness. The move $Ro5^*$ is slightly different but equivalent modulo the $Ro1$ and $Ro2$ moves, to the usual $Ro5$ move.

If X is a set of surface links let $\mathcal{B}(X)$ be the set of broken surface diagrams corresponding to links in X with generic projections, and let $\mathcal{B}^b(X) \subset \mathcal{B}(X)$ be the subset of branch-free broken surface diagrams.

△

Theorem 2.2 (Roseman [R]). *If $D, D' \in \mathcal{B}(\mathcal{L})$ represent isotopic surface links then D and D' are related by a finite sequence of the moves in Figure 3 and equivalences in \mathbb{R}^3 .*

Remark 2.3. In the paper of Bar-Natan, Fulman and Kauffman [BKF] there is a proof of the well-known fact that all spanning-surfaces of a classical link in \mathbb{R}^3 are tube-equivalent. In that proof, link projections are used to construct Seifert surfaces via Seifert's algorithm, and the main result is deduced by observing how the constructed Seifert surfaces change when Reidemeister moves are performed on the link projection.

We wish to approach the proofs of Theorems 2.9 and 4.4 in a similar manner. We will assign various structures (a set of branch-free broken surface diagrams in the case of Theorem 2.9 and an ab -surface in the case of Theorem 4.4) to a broken surface diagram and observe how

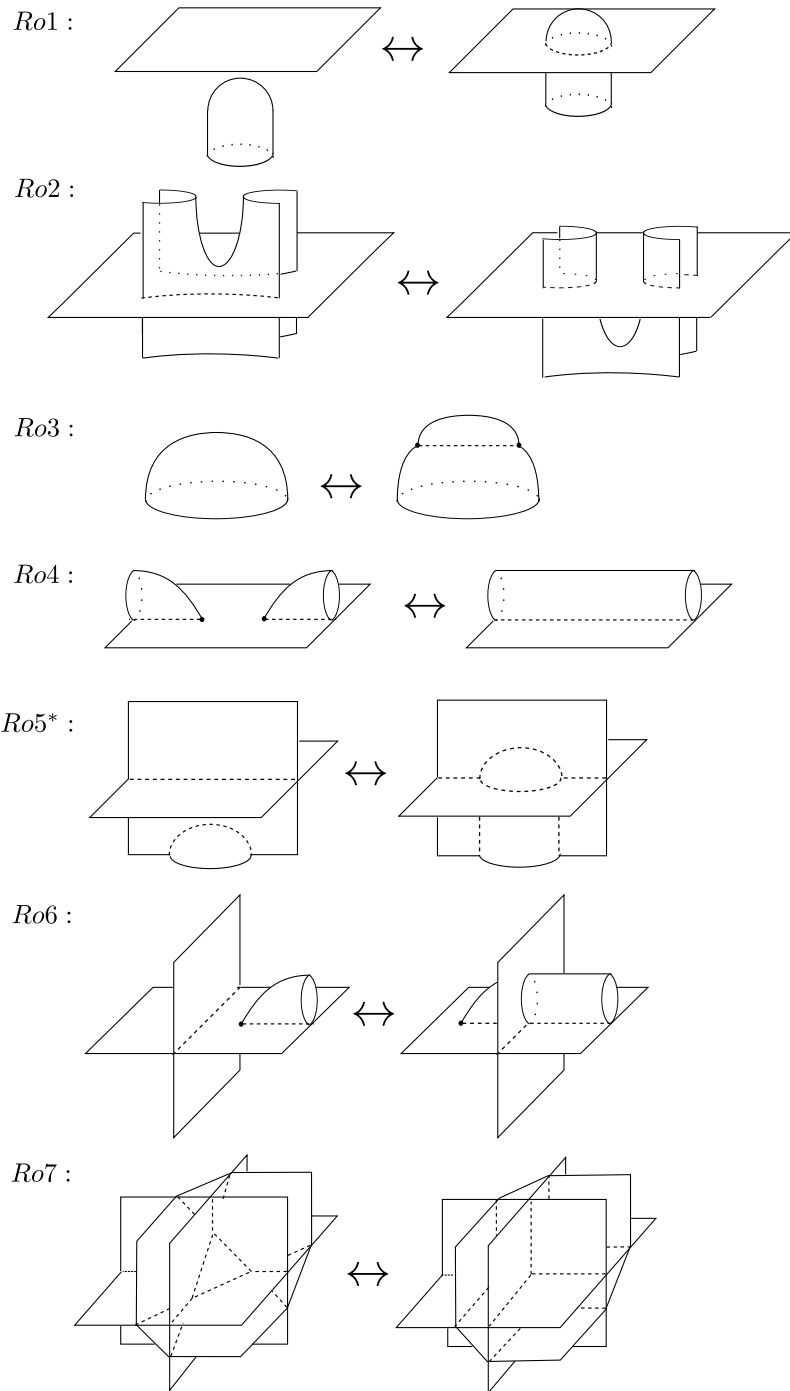


FIGURE 3. Roseman moves on broken surface diagrams, with crossing information suppressed.

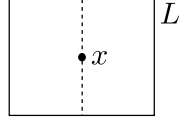


FIGURE 4. The set $\pi^{-1}(\text{sing}(\pi(L))) \subset L$ in the neighbourhood of a branch point x .

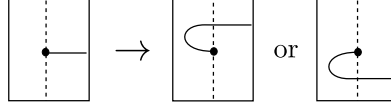


FIGURE 5. Converting an invalid path to a valid path. See Definition 2.4 and Remark 2.5.

these structures change when Roseman moves are performed on the broken surface diagram.

Definition 2.4. Following Carter and Saito [CS], we define a function $\text{cancel} : \mathcal{B}(\mathcal{L}^+) \rightarrow \mathcal{P}(\mathcal{B}^b(\mathcal{L}^+))$ (where $\mathcal{P}(X)$ is the power set of X). Let $D = \pi(L)$ be a broken surface diagram corresponding to a generic projection for some $L \in \mathcal{L}^+$ with components $L = L_1 \cup \dots \cup L_r$. If $x \in L$ is such that $\pi(x)$ is a branch point, then a neighbourhood of $x \in \pi^{-1}(\text{sing}(D)) \subset L$ looks as in Figure 4. Let D_i be the broken surface diagram obtained from D by removing all components except L_i . Each D_i has an equal number of positive and negative branch points, and any pair of positive and negative branch points can be cancelled using an appropriate sequence of $\overrightarrow{Ro6}$ moves followed by an $\overrightarrow{Ro4}$ move.

Specifically, assume D_i has $2b_i \geq 0$ branch points and for $1 \leq i \leq r$ and $1 \leq j \leq b_i$ let $p_{i,j}, q_{i,j} \subset L_i$ be the distinct points such that $\pi(p_{i,j})$ and $\pi(q_{i,j})$ are positive and negative branch points in D . A pair $(T = \{\tau_i\}_{1 \leq i \leq r}, V = \bigcup_{1 \leq i \leq r, 1 \leq j \leq b_i} v_{i,j})$ is **valid** if:

- each τ_i is a permutation of $\{1, 2, \dots, b_i\}$,
- $V \subset L_i$ is a disjoint union of embedded oriented compact intervals $v_{i,j}$ with endpoints $\partial v_{i,j} = \{p_{i,j}, q_{i,\tau_i(j)}\}$ and the orientation going from $p_{i,j}$ to $q_{i,\tau_i(j)}$,
- $V \cap \pi^{-1}(\text{sing}(D)) \subset L$ is transverse in L and $\pi(V)$ is embedded in \mathbb{R}^3 ,
- after performing the sequence of $\overrightarrow{Ro6}$ moves pushing $p_{i,j}$ along $v_{i,j}$ through all intersections in $v_{i,j} \cap \pi^{-1}(\text{sing}(D))$, it is possible to perform a final $\overrightarrow{Ro4}$ move cancelling $p_{i,j}$ and $q_{i,\tau_i(j)}$.

The set $\text{cancel}(D)$ contains a broken surface diagram $E(T, V)$ for each valid pair (T, V) , obtained by actually performing on D the aforementioned $\overrightarrow{Ro6}$ moves along each $v_{i,j}$ ending in an $\overrightarrow{Ro4}$ move cancelling the branch points $p_{i,j}$ and $q_{i,\tau_i(j)}$. If there are no branch points in D we let $\text{cancel}(D) = \{D\}$. \triangle

Remark 2.5. It is important to note that the final requirement for valid paths in Definition 2.4 is not superfluous. There exist invalid collections of simple paths satisfying the first three requirements. However if a particular path satisfies the first three requirements but not the fourth, then it can be made valid by the modification in Figure 5. This is depicted on the level of broken surface diagrams in Carter and Saito [CS, Figure K and Figure M]. \triangle

Remark 2.6. Figure 6 describes P moves on the valid pairs (T, V) of Definition 2.4. For each move, the diagram $D = \pi(L)$ remains fixed and the set $\pi^{-1}(\text{sing}(D)) \subset L$ thus remains fixed as well. The moves $P2 - P6$ correspond to an isotopy of V rel ∂V in L , interacting with the set $\pi^{-1}(\text{sing}(D))$. The moves $P1$ and $P7$ describe two types of interactions of two simple paths in V . The $P1$ move acts as a transposition on one of the permutations in T . These moves are to be thought of only formally, for the moment. In Lemma 2.7 we will see that the moves $P1 - P6$ correspond naturally to moves $Br1 - Br6$ on broken surface diagrams, described in Figure 7. Note that each Br move does in fact represent an isotopy in \mathbb{R}^4 , since it can be written in terms of Roseman moves, in particular $Ro4$ and $Ro6$ moves. \triangle

Lemma 2.7. *a) If $D \in \mathcal{B}(\mathcal{L}_0)$ then all valid pairs (T, V) as in Definition 2.4 are related by the P moves in Figure 6.*

b) The $Br3$, $Br4$, $Br5$ and $Br6$ moves can be expressed in terms of $Ro1$, $Ro2$, $Ro5^$, $Ro7$ and $Br1$ moves.*

c) If $D \in \mathcal{B}(\mathcal{L}_0)$ and $E_1, E_2 \in \text{cancel}(D)$ then E_1 and E_2 are related by the $Ro1$, $Ro2$, $Ro5^$, $Ro7$, $Br1$ and $Br2$ moves in Figures 3 and 7.*

d) If $D_1, D_2 \in \mathcal{B}(\mathcal{L}_0)$, $E_1 \in \text{cancel}(D_1)$, $E_2 \in \text{cancel}(D_2)$ and D_1 and D_2 are related by a Roseman move then E_1 and E_2 are related by a sequence of $Ro1$, $Ro2$, $Ro5^$, $Ro7$, $Br1$ and $Br2$ moves.*

Proof. a) Assume we have two pairs (T, V) and (T', V') with the same set of permutations $T = T' = \{\tau_i\}_{1 \leq i \leq r}$ and let $V = \{v_{i,j}\}_{1 \leq i \leq r, 1 \leq j \leq b_i}$ and $V' = \{v'_{i,j}\}_{1 \leq i \leq r, 1 \leq j \leq b_i}$. If for some i, j the pairs $v_{i,j}$ and $v'_{i,j}$ do not agree in small enough neighbourhoods of the points $p_{i,j}$ and $q_{i,\tau_i(j)}$ in L , then they can be made to agree using a $P6$ move, due to the fact that the pairs (T, V) and (T, V') satisfy the last property

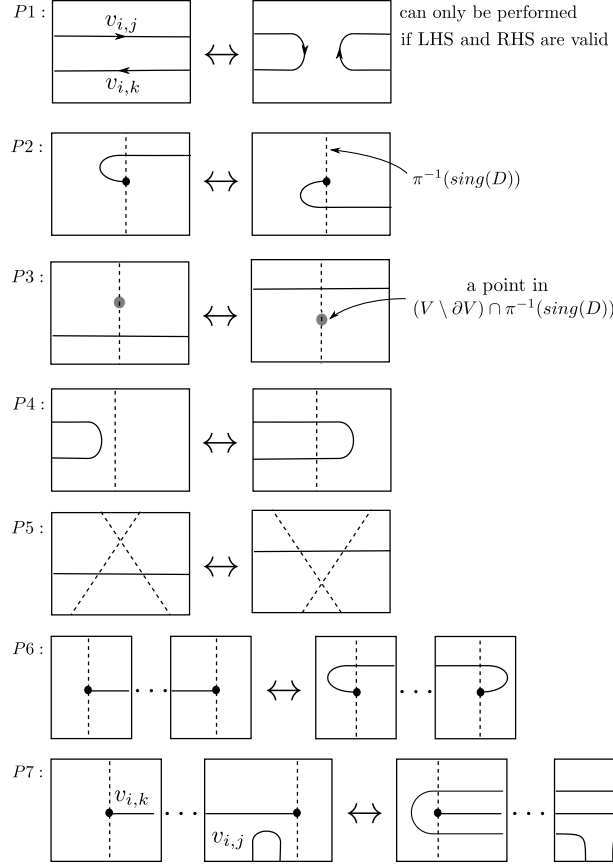


FIGURE 6. Moves on valid pairs (T, V) for some fixed broken surface diagram D . See Remark 2.6.

in Definition 2.4. Thus we assume for each i, j the paths $v_{i,j}$ and $v'_{i,j}$ agree in small enough neighbourhoods of the points $p_{i,j}$ and $q_{i,\tau_i(j)}$ in L . Since each component of L is a sphere, there is an isotopy in L taking $v_{i,j}$ to $v'_{i,j}$ relative to their common boundary $p_{i,j}$ and $q_{i,\tau_i(j)}$. During this isotopy, $v_{i,j}$ does not change in a small enough neighbourhood of its boundary. Such an isotopy can be accomplished using the $P2$, $P3$, $P4$, $P5$, and $P7$ moves. Thus after performing each isotopy for all choices of i, j we get a sequence of P moves taking V to V' .

Consider now the case of two pairs (T, V) and (T', V') with $T \neq T'$. It is enough to assume that all permutations in T agree with their respective permutations in T' except for some two permutations that differ by a transposition. We can induce an arbitrary transposition using a $P1$ move. If we wish to perform a $P1$ move

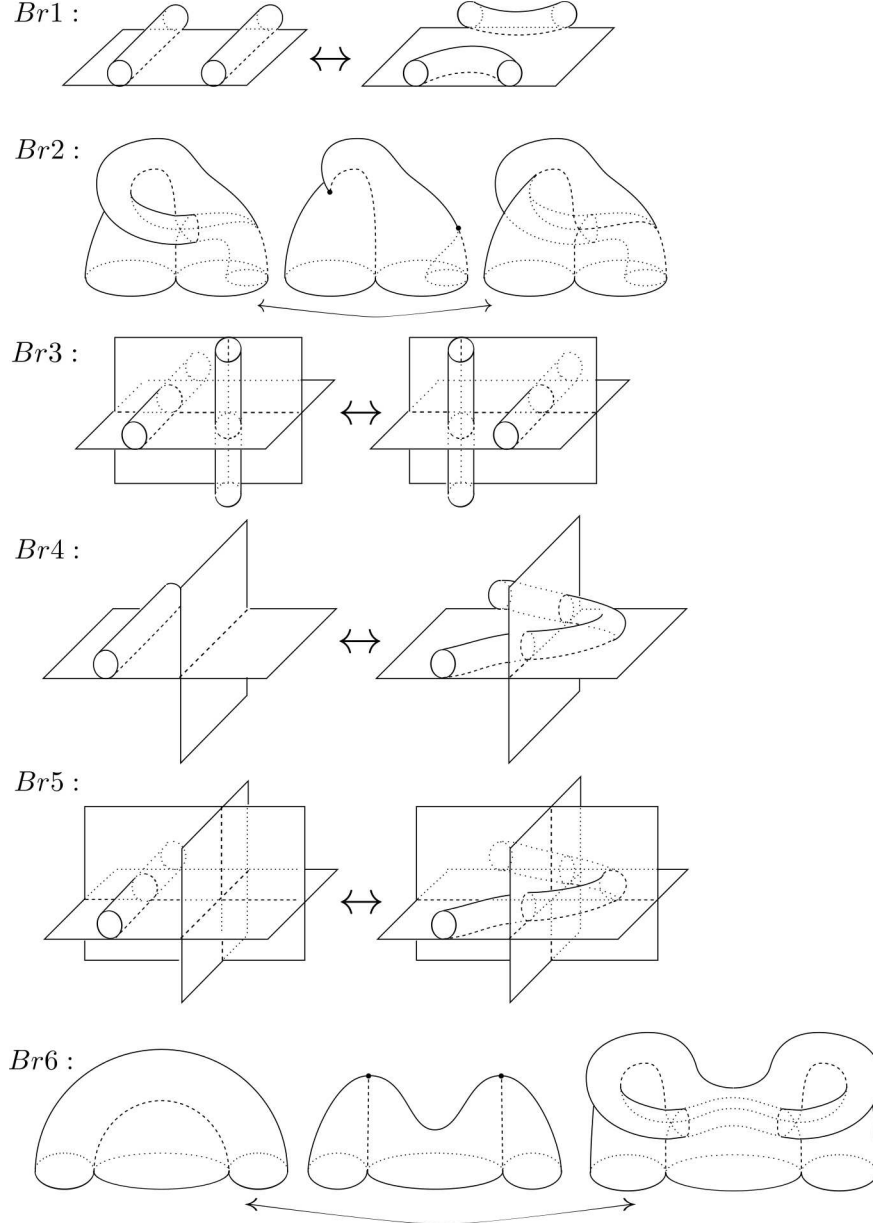


FIGURE 7. *Br* moves for broken surface diagrams, with crossing information suppressed.

between $v_{i,j}$ and $v_{i,k}$ we can use *P4* moves to bring them into the exact form in the left-hand side of the *P1* move. Now if the *P1* move is valid (i.e. lifts to the *Br1* move in Figure 7), we can perform the *P1* move to induce a transposition. If the *P1* move is not valid, it

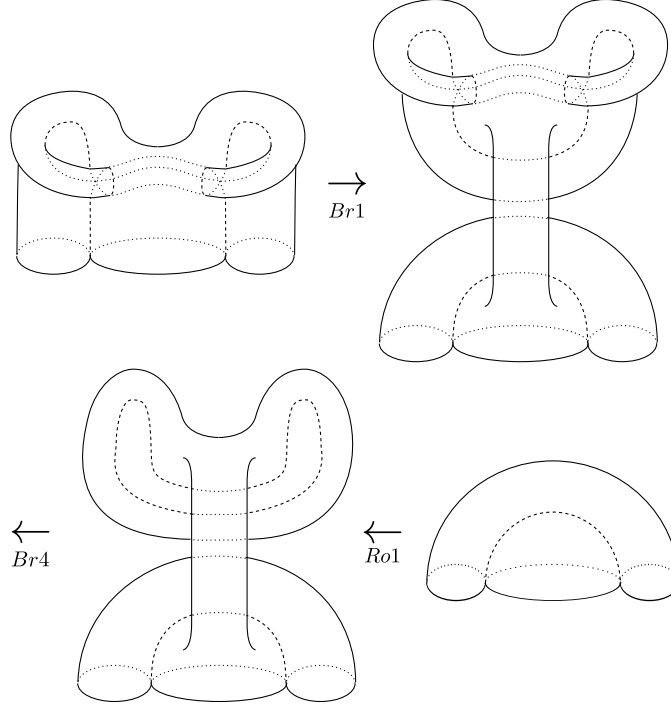


FIGURE 8. Showing that a $Br6$ move can be realized with $Ro1$, $Br1$ and $Br4$ moves.

can be made valid after a $P6$ move along either of the paths. Thus we may induce arbitrary transpositions in the τ_i 's using P moves.

Thus the P moves suffice to connect all valid pairs (T, V) in Definition 2.4.

- b) We leave it to the reader to verify that the $Br3$, $Br4$, and $Br5$ moves can be expressed in terms of $Ro1$, $Ro2$, $Ro5^*$ and $Ro7$ moves. Figure 8 expresses a $Br6$ move using $Ro1$, $Br1$ and $Br4$ moves.
- c) If $E_1, E_2 \in \text{cancel}(D)$ then by Lemma 2.7a their defining valid pairs are related by P moves. The moves $P1$, $P2$, $P3$, $P4$ and $P5$ on valid pairs can be realized directly by the moves $Br1$, $Br2$, $Br3$, $Br4$ and $Br5$ respectively on broken surface diagrams. The move $P6$ can be realized by a sequence involving $Ro1$, $Ro2$, $Ro5^*$, $Ro7$ and $Br6$ moves. The $P7$ move can be realized by a sequence involving $Ro1$, $Ro2$, $Ro5^*$ and $Ro7$ moves. By Lemma 2.7b, the $Br3$, $Br4$, $Br5$ and $Br6$ moves can be expressed in terms of $Ro1$, $Ro2$, $Ro5^*$, $Ro7$ and $Br1$ moves, so we are done.
- d) If D_1 and D_2 are related by an $Ro1$, $Ro2$, $Ro5^*$, or $Ro7$ move, then it is not difficult to see that there are diagrams $F_1 \in \text{cancel}(D_1)$ and

$F_2 \in \text{cancel}(D_2)$ such that F_1 and F_2 are related by an $Ro1$, $Ro2$, $Ro5^*$, or $Ro7$ move. By Lemma 2.7c, E_1 is related to F_1 and E_2 is related to F_2 by a sequence of $Ro1$, $Ro2$, $Ro5^*$, $Ro7$, $Br1$ and $Br2$ moves. Thus there is a sequence of $Ro1$, $Ro2$, $Ro5^*$, $Ro7$, $Br1$ and $Br2$ moves relating E_1 to E_2 .

If an $\overrightarrow{Ro3}$ move takes D_1 to D_2 then there are diagrams $F_1 \in \text{cancel}(D_1)$ and $F_2 \in \text{cancel}(D_2)$ with a $\overleftarrow{Ro1}$ move taking F_1 to F_2 (note that the two branch points created by the $\overrightarrow{Ro3}$ move can be paired in two obvious ways. One way can be cancelled by an $\overleftarrow{Ro1}$ move and the other can be cancelled by $\overleftarrow{Ro1}$, $Br1$ and $Br4$ moves, similar to Figure 8). Thus by Lemma 2.7c again there is a sequence of $Ro1$, $Ro2$, $Ro5^*$, $Ro7$, $Br1$ and $Br2$ moves relating E_1 to E_2 .

If D_1 and D_2 are related by an $Ro4$ or an $Ro6$ move, then $\text{cancel}(D_1) \subset \text{cancel}(D_2)$ or $\text{cancel}(D_2) \subset \text{cancel}(D_1)$ and we rely on Lemma 2.7c once more.

□

Remark 2.8. It should be noted that the proof of Lemma 2.7a relies on the fact that in a sphere, any two simple paths with a common boundary are isotopic relative to their common boundary. \triangle

Now we can prove a branch-free version of Roseman's theorem.

Theorem 2.9. *If $D_1, D_2 \in \mathcal{B}^b(\mathcal{L}_0)$ are related by a finite sequence of Roseman moves then they are related by a finite sequence of $Ro1$, $Ro2$, $Ro5^*$, $Ro7$, $Br1$ and $Br2$ moves.*

Proof. Let $D_1 = E_1, \dots, E_n = D_2$ be a sequence of broken surface diagrams with E_{i+1} related to E_i by a Roseman move for $1 \leq i < n$. Choose $F_i \in \text{cancel}(E_i)$ for $1 \leq i \leq n$. By Lemma 2.7d, F_{i+1} is related to F_i by a sequence of $Ro1$, $Ro2$, $Ro5^*$, $Ro7$, $Br1$ and $Br2$ moves, for $1 \leq i < n$. Since $\text{cancel}(E_1) = \text{cancel}(D_1) = \{D_1\}$ and $\text{cancel}(E_n) = \text{cancel}(D_2) = \{D_2\}$, we are done. \square

3. MARKED GRAPHS AND ab -SURFACES

Lomonaco [L] and Yoshikawa [Yo] described another method of representing surface links via certain 4-regular rigid vertex graphs in \mathbb{R}^3 . Definition 3.1 reviews marked graphs and the Yoshikawa moves on marked graph diagrams. Definition 3.2 concerns ab -surfaces and ab -moves. Definition 3.3 presents the *thicken* and *cap* functions. Lemma 2.7 describes some key properties pertaining to marked graphs, ab -surfaces and the *thicken* and *cap* functions.

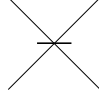
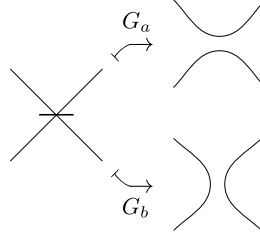


FIGURE 9. A 4-valent rigid vertex with a marker.


 FIGURE 10. The links $G_a, G_b \subset \mathbb{R}^3$ obtained by resolving all marked vertices in a marked graph G .

Definition 3.1. A **marked graph** $G \subset \mathbb{R}^3$ is a 4-regular rigid vertex graph (with some components possibly having no vertices) where:

- the regions in a rigid neighbourhood of each vertex are given a checkboard coloring, which is usually indicated with a line segment, or marker, as in Figure 9,
- the links $G_a, G_b \subset \mathbb{R}^3$ obtained by resolving each marked vertex as in Figure 10, are trivial.

We consider two marked graphs **equivalent** if they differ by the action of $\text{Diff}_0^+(\mathbb{R}^3)$ in a way that preserves rigid neighbourhoods of their vertices, and have agreeing markers. Let \mathcal{M} be the set of marked graphs. One may project a marked graph to \mathbb{R}^2 and obtain a **marked graph diagram**. Figure 11 describes the Yoshikawa moves on marked graph diagrams. The *Type I* moves do not change the equivalence class of the marked graph in \mathbb{R}^3 . The results in Kauffman [Kau] show that these first five moves do in fact generate the equivalence relation on marked graph diagrams coming from the action of $\text{Diff}_0^+(\mathbb{R}^3)$ on marked graphs in \mathbb{R}^3 . The *Type II* moves differ from *Type I* moves in that they are defined for marked graphs in \mathbb{R}^3 , not just marked graph diagrams in \mathbb{R}^2 . \triangle

Definition 3.2. An ***ab*-surface** $R \subset \mathbb{R}^3$ is a compact not necessarily connected not necessarily orientable surface with boundary where:

- each boundary component of R is assigned a label from the set $\{a, b\}$,

Type I

$$\Omega_1 : \text{diagram} \leftrightarrow \text{diagram} \leftrightarrow \text{diagram}$$

$$\Omega_2 : \text{diagram} \leftrightarrow \text{diagram}$$

$$\Omega_3 : \text{diagram} \leftrightarrow \text{diagram}$$

$$\Omega_4 : \text{diagram} \leftrightarrow \text{diagram}$$

$$\Omega'_4 : \text{diagram} \leftrightarrow \text{diagram}$$

$$\Omega_5 : \text{diagram} \leftrightarrow \text{diagram}$$

Type II

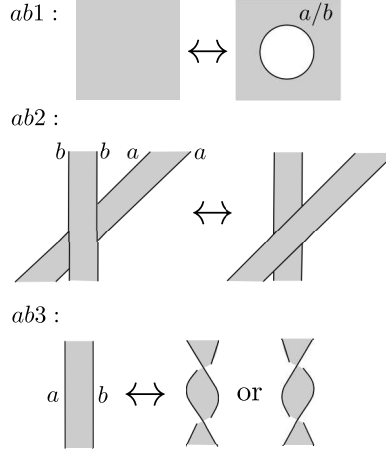
$$\Omega_{6a} : \text{diagram} \leftrightarrow \text{diagram}$$

$$\Omega_{6b} : \text{diagram} \leftrightarrow \text{diagram}$$

$$\Omega_7 : \text{diagram} \leftrightarrow \text{diagram}$$

$$\Omega_8 : \text{diagram} \leftrightarrow \text{diagram}$$

FIGURE 11. Yoshikawa moves on marked graph diagrams.

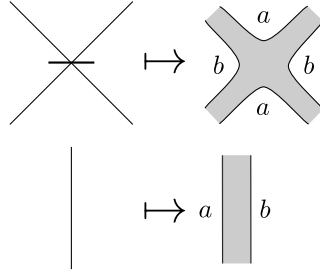
FIGURE 12. ab -moves on ab -surfaces.

- each component of R contains at least one a -labelled and at least one b -labelled boundary component,
- the a -labelled and b -labelled boundary links $\partial_a R, \partial_b R \subset \mathbb{R}^3$ are trivial.

We consider two ab -surfaces **equivalent** if they differ by the action of $\text{Diff}_0^+(\mathbb{R}^3)$ and have matching labellings of their boundaries. Let \mathcal{S} be the set of ab -surfaces, \mathcal{S}^+ the subset of orientable ab -surfaces, and \mathcal{S}_0 the subset of orientable ab -surfaces in which each component has genus 0. We will refer to the moves on ab -surfaces in Figure 12 as **ab -moves**. \triangle

Definition 3.3. The function $\text{thicken} : \mathcal{M} \rightarrow \mathcal{S}/ab3$, mapping a marked graph to an ab -surface up to $ab3$ moves, is given in Figure 13. Since this is defined locally on the edges and marked vertices of a marked graph, we must glue the final result in a way that preserves the a/b -labels coming from marked vertices, and this can only be done up to $ab3$ moves. Note that $G_a = \partial_a(\text{thicken}(G))$ and $G_b = \partial_b(\text{thicken}(G))$ holds for any 4-regular rigid vertex graph G that satisfies the first condition in Definition 3.1. For $R \in \mathcal{S}$ denote by $\text{thicken}^{-1}(R)$ the set of all $G \in \mathcal{M}$ with $R \in \text{thicken}(G)$.

There is a function $\text{cap} : \mathcal{S} \rightarrow \mathcal{L}/\text{Diff}_0^+(\mathbb{R}^4)$ defined as follows. Given an arbitrary ab -surface R , the boundary of R forms two trivial links $\partial_a R, \partial_b R$. Let $\{D_a^i\}_i$ and $\{D_b^j\}_j$ each be a disjoint union of embedded disks in \mathbb{R}^3 with $\cup_i D_a^i = \partial_a R$ and $\cup_j D_b^j = \partial_b R$ respectively. We obtain a surface link by pushing the interiors of the disks D_a^i into \mathbb{R}_1^3 and the interiors of the disks D_b^j into \mathbb{R}_{-1}^3 , where we use the notation $\mathbb{R}_{u_0}^3 =$

FIGURE 13. The map $thicken : \mathcal{M} \rightarrow \mathcal{S}/ab3$.

$\{(x, y, z, u) \in \mathbb{R}^4 : u = u_0\}$. The resulting surface link isotopy class, which we denote $cap(R)$, is independent of the choice of disk systems $\{D_a^i\}_i$ and $\{D_b^j\}_j$ in \mathbb{R}^3 , see Kamada [Kam, Proposition 8.6]. Due to this definition, it is reasonable to think of the labels as shorthand for a and b as “above” and “below” (with respect to the u -coordinate).

Given a marked graph $G \in \mathcal{M}$, the surface link isotopy class associated to G is the isotopy class of $cap(R)$ for any $R \in thicken(G)$. We will see in Lemma 3.4d that the cap function is invariant under all ab -moves, so in particular the isotopy class of $cap(R)$ is the same for all $R \in thicken(G)$, since all such R are related by $ab3$ moves. \triangle

- Lemma 3.4.** a) If $R \in \mathcal{S}_0$ then $thicken^{-1}(R)$ is non-empty and any two graphs in $thicken^{-1}(R)$ are related by Ω_7 moves.
b) If $G_1, G_2 \in \mathcal{M}$ are related by Type II Yoshikawa moves and $R_1 \in thicken(G_1)$ and $R_2 \in thicken(G_2)$ then R_1 and R_2 are related by ab -moves.
c) If $R_1, R_2 \in \mathcal{S}_0$ are related by ab -moves and $G_1 \in thicken^{-1}(R_1)$ and $G_2 \in thicken^{-1}(R_2)$ then G_1 and G_2 are related by Type II Yoshikawa moves.
d) If $R_1, R_2 \in \mathcal{S}$ are related by ab -moves then $cap(R_1) = cap(R_2)$.

Proof. a) If $G \in thicken^{-1}(R)$ then G is equivalent to a graph embedded in R . Thus there is no loss in generality restricting ourselves to marked graphs G with $G \subset R$. There is also no loss in generality in assuming R is connected.

Assume R has a -boundary components a_1, \dots, a_v and b -boundary components b_1, \dots, b_f for $v, f \geq 1$. Let $c_1, \dots, c_v \subset R$ be disjoint simple closed curves with c_i parallel to a_i . Let $p_1, \dots, p_{v+f-2} \subset R$ be a collection of disjoint simple arcs, whose interiors are disjoint from all curves c_i , with $\partial p_j \subset \bigcup_i c_i$ and such that each component of $R \setminus \left(\bigcup_i c_i \cup \bigcup_j p_j \right)$ is an annulus containing one unlabelled boundary

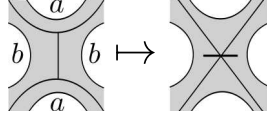


FIGURE 14. Constructing a marked graph in $\text{thicken}^{-1}(R)$ from an a -system in Lemma 3.4a.

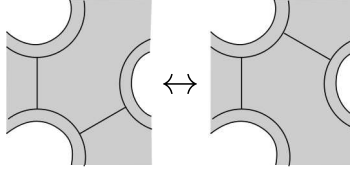


FIGURE 15. The slide move for a -systems in R in Lemma 3.4a.

component and one labelled boundary component. One may view the curves c_i and arcs p_j as specifying a planar graph, with vertices the curves c_i , faces the b -labelled boundary components, and edges the arcs p_j .

We call such a collection of simple curves and arcs up to isotopies in R (i.e. up to the action of $\text{Diff}_0^+(R)$), an **a -system**. We may construct a marked graph $G \in \text{thicken}^{-1}(R)$ with $G \subset R$ from an a -system via the transformation in Figure 14. Conversely, given a marked graph $G \subset R$, the inverse of the operation in Figure 14, creates an a -system. The Ω_7 move on marked graphs in R corresponds to the slide move in Figure 15 on a -systems. One can use slide moves to bring any a -system into a form where there exist two open intervals $U, V \subset c_1$ with $U \cap V = \emptyset$ such that $v - 1$ of the paths have one endpoint in U and one endpoint on c_i for $1 < i \leq v$ and the remaining $f - 1$ paths have both endpoints adjacent to each other in V . The claim that any two such a -systems are related by slide moves can be deduced from Kamada [Kam, Proposition 2.14].

Note that the set of a -systems up to the action of $\text{Diff}_0^+(R)$ is infinite, if R has four or more boundary components. However in this case $\text{thicken}^{-1}(R)$ may still be finite if R admits many symmetries in \mathbb{R}^3 , say if $R \subset \mathbb{R}^2$ (in which case all of the a -systems convert into one of finitely many marked graphs up to the action of $\text{Diff}_0^+(\mathbb{R}^3)$).

- b) If G_1 is related to G_2 by an Ω_{6a} or Ω_{6b} move then R_1 and R_2 are related by $ab1$ and $ab3$ moves. If G_1 and G_2 are related by an Ω_7 move then $\text{thicken}(G_1) = \text{thicken}(G_2)$ and R_1 and R_2 are related by

$ab3$ moves. If G_1 and G_2 are related by an Ω_8 move then R_1 and R_2 are related by $ab2$ and $ab3$ moves.

- c) We consider each ab -move separately. Assume an $\overrightarrow{ab1}$ move takes R_1 to R_2 and creates an a -labelled boundary component. Then one may obtain an a -system for R_2 from any a -system for R_1 by connecting an extra path to the new simple closed curve parallel to the new a -labelled boundary component. This corresponds to an $\overrightarrow{\Omega_{6a}}$ move. Thus G_1 and G_2 are related by Ω_7 and Ω_{6a} moves. If the $ab1$ move involves a b -labelled boundary component the same reasoning may be used except with the dual notion of b -systems, and G_1 and G_2 will be related by Ω_7 and Ω_{6b} moves.

Assume now R_1 and R_2 are related by an $ab2$ move. Any $ab2$ move determines two disjoint simple paths $p, q \subset R_1$ with $\partial p \subset \partial_a R_1$ and $\partial q \subset \partial_b R_1$. If either path cobounds with a segment of ∂R_1 an embedded disk in R_1 , then the $ab2$ move can be realized as an equivalence of ab -surfaces in \mathbb{R}^3 . Thus assume neither path cobounds such a disk. First consider the case where p, q are in different components of R_1 . One can find an a -system in the component containing p such that p is one of the paths of the a -system and similarly one can find a b -system in the component containing q such that q is one of the paths of the b -system. By finding a or b -systems in the remaining components of R_1 , we obtain a marked graph $G'_1 \in \text{thicken}(R_1)$ for which the $ab2$ move corresponds to an Ω_8 move taking G'_1 to some $G'_2 \in \text{thicken}(R_2)$. Thus G_1 and G_2 are related by Ω_7 and Ω_8 moves. Now consider the case where p, q are in the same component of R_1 . Since we assumed neither p, q cobounds with ∂R_1 an embedded disk in R_1 , the component of R_1 containing p, q must have at least two a -labelled and two b -labelled boundary components. We must find an a -system for this component of R_1 , such that p is one of the paths of the a -system and q is one of the paths of the dual b -system (any a -system induces a unique dual b -system and vice versa). This amounts to finding an a -system for which p is one of the paths of the a -system and q transversely intersects some other path in this a -system (not p) exactly once. It is not difficult to see that any such path along with p , can be extended to an a -system so we are done. As before, we obtain a marked graph $G'_1 \in \text{thicken}(R_1)$ for which the $ab2$ move corresponds to an Ω_8 move taking G'_1 to some $G'_2 \in \text{thicken}(R_2)$. Thus G_1 and G_2 are related by Ω_7 and Ω_8 moves.

Assume finally that R_1 and R_2 are related by an $ab3$ move. The $ab3$ move determines a simple compact interval p in R_1 with one endpoint in $\partial_a R_1$ and the other in $\partial_b R_1$. We can readily find an

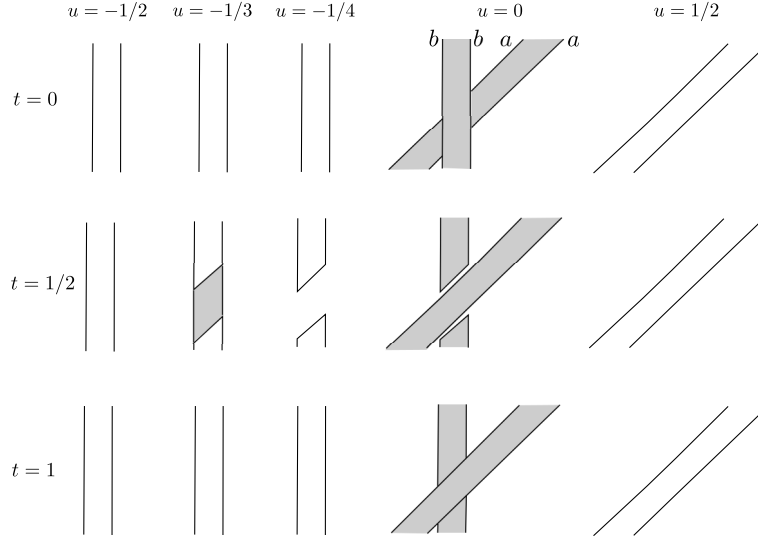


FIGURE 16. An isotopy between $\text{cap}(R_1)$ and $\text{cap}(R_2)$ when R_1 and R_2 are related by an $ab2$ move.

a -system in R_1 for which all paths in the system are disjoint from p . This a -system gives rise to a marked graph G_0 for which $R_1, R_2 \in \text{thicken}(G_0)$. Thus G_1 and G_2 are related to G_0 by Ω_7 moves and hence are related to each other by Ω_7 moves.

- d) For the $ab1$ move this should be clear. For the $ab2$ and $ab3$ moves, the isotopies in \mathbb{R}^4 connecting $\text{cap}(R_1)$ and $\text{cap}(R_2)$ are given in Figures 16 and 17.

□

Remark 3.5. It is likely that all of Lemma 3.4 can be generalized to ab -surfaces in \mathcal{S}^+ , without restricting to \mathcal{S}_0 in some instances as we have done.

4. A MAP FROM BROKEN SURFACE DIAGRAM TO ab -SURFACES

In Definition 4.1 we describe the function *perforate*, assigning to any branch-free broken surface diagram an ab -surface. In Lemma 4.3 we prove some properties of the *perforate* function and observe how it behaves when $Ro1$, $Ro2$, $Ro5^*$, $Ro7$, $Br1$ and $Br2$ moves are performed on the input. In Theorem 4.4 we prove the main result, that two marked graphs representing isotopic 2-links are related by a sequence of *Type II* Yoshikawa moves.

Definition 4.1. We define the function $\text{perforate} : \mathcal{B}^b(\mathcal{L}) \rightarrow \mathcal{S}$ by the transformations in Figure 18, with a caveat. If the surface obtained via

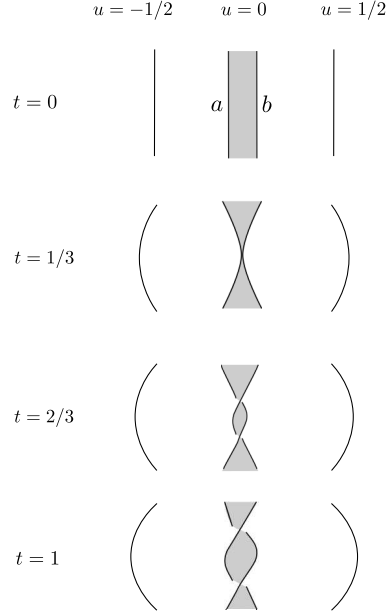


FIGURE 17. An isotopy between $\text{cap}(R_1)$ and $\text{cap}(R_2)$ when R_1 and R_2 are related by an $ab3$ move.

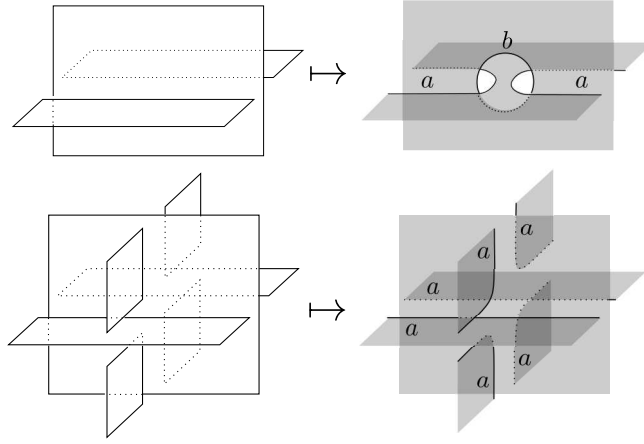


FIGURE 18. The function $\text{perforate} : \mathcal{B}^b(\mathcal{L}) \rightarrow \mathcal{S}$.

Figure 18 has no a -labelled (resp. b -labelled) boundary components, we add arbitrarily a small b -labelled (resp. a -labelled) boundary component to fulfil the second property of Definition 3.2. However we will often not bother to draw these extra components. \triangle

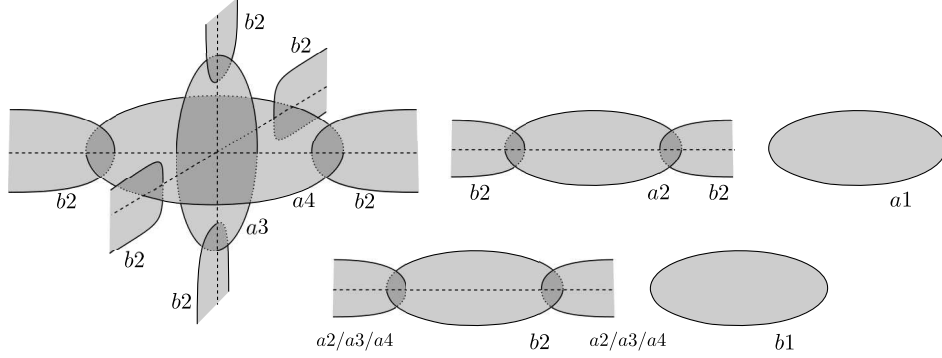


FIGURE 19. Types of disks in the proof of Lemma 4.3a and 4.3b. Note that disks of type $a3$ and $a4$ are dependent and can only occur together.

Remark 4.2. If $L \in \mathcal{L}$ is such that $\pi(L) \in \mathcal{B}^b(\mathcal{L})$, the set $\text{sing}(\pi(L))$ is a 6-regular graph, possibly with some components having no vertices. Each edge or closed component gives rise to one b -labelled boundary component in $\text{perforate}(\pi(L))$. The surface $\text{perforate}(\pi(L))$ is indeed an ab -surface. The b -labelled boundary $\partial_b(\text{perforate}(\pi(L)))$ forms an unlink and the existence of a b -labelled boundary component at each edge forces $\partial_a(\text{perforate}(\pi(L)))$ to form an unlink as well. The isotopy class $\text{cap}(\text{perforate}(\pi(L)))$ agrees with the isotopy class of L . Note also that the ab -surface $\text{perforate}(D)$ for any $D \in \mathcal{B}^b(\mathcal{L})$ induces the 0-framing on each of its boundary components. \triangle

Lemma 4.3. a) If $R \in \mathcal{S}$ and there exist systems $\{D_a^i\}_i$ and $\{D_b^j\}_j$ of disks in \mathbb{R}^3 such that:

- $\cup_i D_a^i = \partial_a R$ and $\cup_j D_b^j = \partial_b R$,
- each disk is embedded and is of one of the types in Figure 19, based on its intersections with other disks and R (which is not depicted in the figure),

then R is related by ab -moves to a surface of the form $\text{perforate}(D)$ for some $D \in \mathcal{B}^b(\mathcal{L})$.

- b) If $R \in \mathcal{S}_0$ then R is related by ab -moves to an ab -surface of the form $\text{perforate}(D)$ for some $D \in \mathcal{B}^b(\mathcal{L}_0)$.
- c) If two broken surface diagrams $D, D' \in \mathcal{B}^b(\mathcal{L})$ are related by an $Ro1$, $Ro2$, $Ro5^*$, $Ro7$, $Br1$ or $Br2$ move then $\text{perforate}(D)$ and $\text{perforate}(D')$ are related by ab -moves.

Proof. a) Each boundary component of the ab -surface R is an unknot and has an induced framing from the embedding $R \subset \mathbb{R}^3$. Due to the way R intersects the described systems of disks, it must be the case

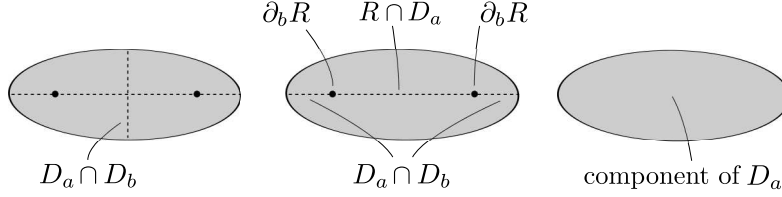


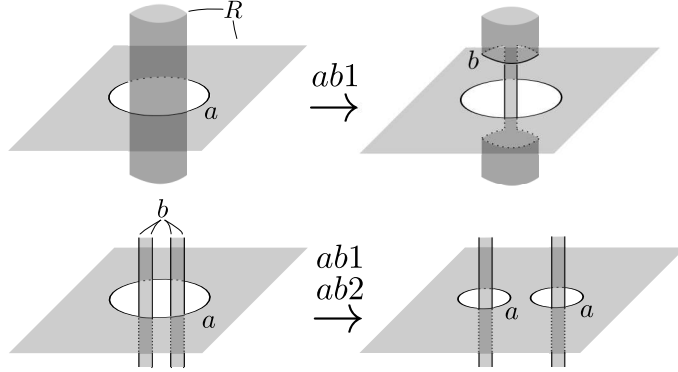
FIGURE 20. Three desirable possibilities for a component of D_a intersecting $R \cup D_b$ in Lemma 4.3a.

that R induces the 0-framing on each of its boundary components. The union $D = R \cup \bigcup_i D_a^i \cup \bigcup_j D_b^j$ is a generic projection with no branch points. The surface R is nearly in the form $\text{perforate}(D)$, we must only eliminate all boundary components bounding disks of type $a1$, $b1$ or $a2$. If there are disks of type $a1$ or $b1$, then those boundary components of R should be eliminated with $\overleftarrow{ab1}$ moves. If there are disks of type $a2$ then those boundary components may be eliminated with $ab2$ and $\overleftarrow{ab1}$ moves. Once this has been done, we have $R = \text{perforate}(D)$.

- b) First we show that we can use $ab3$ moves to ensure each boundary component of R is 0-framed. It is enough to prove this when R is connected. By the second property of Definition 3.2, R has at least one a -labelled and one b -labelled boundary component. We can use $ab3$ moves to ensure that all framings are 0 except for one a -labelled boundary component. By Lemma 3.4a there is $G \in \mathcal{M}$ with $G \subset R$ and $\text{thicken}(G) = R$. The graph G is planar, and we can unknot it to some plane graph in \mathbb{R}^2 , while preserving the framings. In this form it is easy to see that this final a -labelled boundary component must automatically be 0-framed, thus it must be 0-framed in R as well.

Let $\{D_a^i\}_i$ and $\{D_b^j\}_j$ be systems of disks as in the definition of the cap function. Note that by the definition of cap each union $D_a = \bigcup_i D_a^i$ and $D_b = \bigcup_j D_b^j$ is embedded in \mathbb{R}^3 . Since each component of ∂R is 0-framed, we may assume that $R \cap D_a$ and $R \cap D_b$ are contained in the interiors of D_a and D_b . We perturb D_a and D_b so that $R \cap D_a$, $R \cap D_b$ and $D_a \cap D_b$ are transverse in \mathbb{R}^3 . We also assume that $(R \cap D_a) \cap (D_a \cap D_b)$ is transverse in D_a , etc. Now with such a system of disks, $R \cup D_a \cup D_b \subset \mathbb{R}^3$ is a generic projection of $\text{cap}(R)$ with no branch points. Ultimately, we would like to find systems of disks as in Lemma 4.3a.

Our next goal is to make a series of adjustments to R , so that each component of D_a intersects $R \cup D_b$ in one of the three ways in Figure

FIGURE 21. Adjusting R in the proof of Lemma 4.3a.

20. We may use $\overrightarrow{ab1}$ moves to add extra b -labelled boundary components so that $R \cap D_a$ contains no closed curves. Specifically, for each closed curve in this intersection we create a b -labelled boundary component in a neighbourhood of some point on the curve. The closed curve then is replaced with a compact interval, as for example in the first row of Figure 21. We may use $\overrightarrow{ab1}$ moves to add extra a -labelled boundary components, followed by $ab2$ moves so that each disk in D_a intersects R in exactly one compact interval, as in the second row of Figure 21. We make further adjustments by shrinking each boundary component in $\partial_a R = \partial D_a$ small enough (via an equivalence in \mathbb{R}^3 of the surface R), so that each component of D_a intersects $R \cup D_b$ as in the top-left of Figure 22, possibly with more transverse intersections from $(R \cap D_a) \cap (D_a \cap D_b)$ (the figure depicts two such intersections). We then may use a series of $ab1$ and $ab2$ moves as in Figure 22, to achieve our stated goal of having each component of D_a intersect $R \cup D_b$ in one of the three ways in Figure 20.

The union $R \cup D_a \cup D_b$ remains a generic projection with no branch points, and the disk systems D_a and D_b remain embedded in \mathbb{R}^3 . Each triple point of $R \cup D_a \cup D_b$ arises from a component of D_a as in the top-left of Figure 20. We perform $ab1$ moves to add an a -labelled boundary component at each triple point, as in Figure 23. By now, for each j we have that $R \cap D_b^j$ is a disjoint union of simple closed curves and compact intervals with endpoints in $\partial_a R$. Each compact interval in $R \cap D_b^j$ is properly contained in an edge of $\text{sing}(R \cup D_a \cup D_b)$ and each simple closed curve in $R \cap D_b^j$ is disjoint from any triple point in $\text{sing}(R \cup D_a \cup D_b)$ and remains a split simple closed curve in $\text{sing}(R \cup D_a \cup D_b)$. We repeat the steps of Figure 21

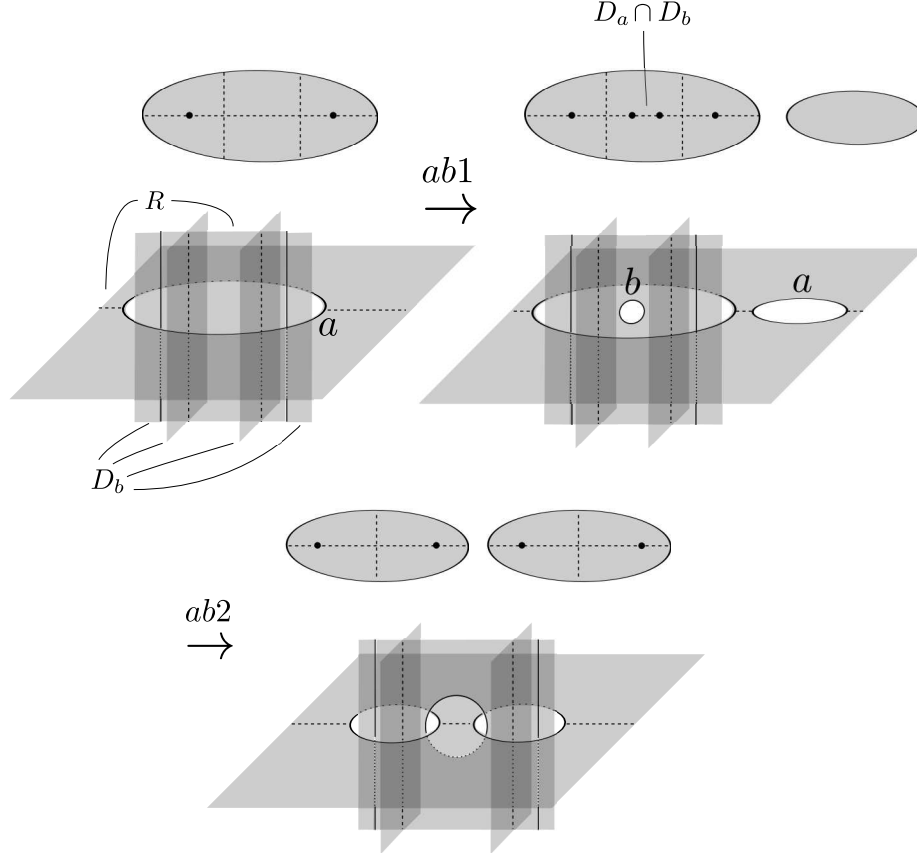


FIGURE 22. Further adjustments of R in Lemma 4.3a, so that each component of D_a intersects $R \cup D_b$ in one of the three ways in Figure 20.

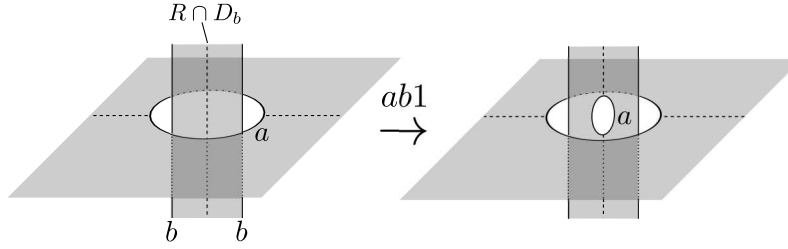
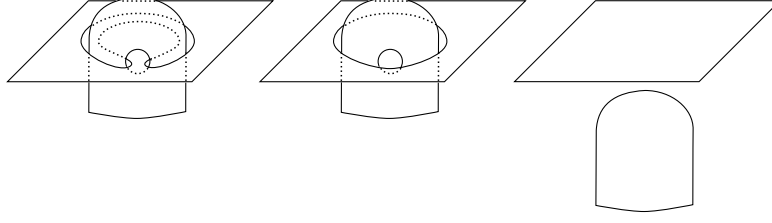


FIGURE 23. Adding an a -labelled boundary component at a triple point.

for the disks D_b^j with the roles of a and b reversed. At this point, the systems $\{D_a^i\}_i$ and $\{D_b^j\}_j$ of disks satisfy the conditions of Lemma 4.3a and we are done.

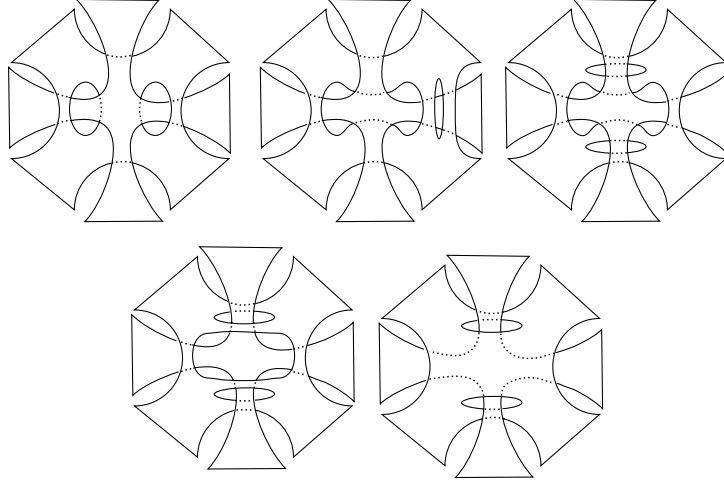
FIGURE 24. Checking the *Ro1* move in Lemma 4.3c.

- c) We must check that each of the mentioned moves on branch-free broken surface diagrams can be realized by *ab*-moves on their images under the *perforate* function. See Figures 24, 25, 26 and 29 for the *Ro1*, *Ro2*, *Ro5** and *Br2* moves. The reader should verify that the transitions in each figure are achievable by *ab*-moves and equivalences in \mathbb{R}^3 . For the *Ro7* move, Figure 27 shows the necessary changes to the bottommost sheet, which will have only *b*-labelled boundary components. One may use *ab1* and *ab2* moves to merge all *b*-labelled boundary components into a single boundary component, the sheet can then be pushed via an equivalence in \mathbb{R}^3 past the triple point (or where it normally would be), and then the *b*-labelled boundary components can be restored with *ab1* and *ab2* moves. In the figure, the dashed line indicates where the other three sheets would intersect the bottommost sheet in the broken surface diagram.

Instead of the *Br2* move, in Figure 28 we check a move that is equivalent to *Br2* modulo *Ro1*, *Ro2*, *Ro5**, *Ro7* and *Br1* moves. \square

Theorem 4.4. *If $G_1, G_2 \in \mathcal{M}$ are such that $\text{cap}(\text{thicken}(G_1))$ and $\text{cap}(\text{thicken}(G_2))$ are isotopic 2-links, then G_1 and G_2 are related by a sequence of Type II Yoshikawa moves.*

Proof. Let $R_1 \in \text{thicken}(G_1)$ and $R_2 \in \text{thicken}(G_2)$. By Lemma 4.3b there exists an *ab*-surface R'_i related by *ab*-moves to R_i so that R'_i is of the form *perforate*(D_i) for some $D_i \in \mathcal{B}^b(\mathcal{L}_0)$ for $i = 1, 2$. Since D_1 represents a 2-link isotopic to the 2-link represented by D_2 , by Roseman's theorem there is a sequence of Roseman moves from D_1 to D_2 . By Theorem 2.9 there is a sequence of *Ro1*, *Ro2*, *Ro5**, *Ro7*, *Br1* and *Br2* moves taking D_1 to D_2 . By Lemma 4.3c there is a sequence of *ab*-moves taking R'_1 to R'_2 . Thus there is a sequence of *ab*-moves taking R_1 to R_2 . By Lemma 3.4c there is a sequence of Type II Yoshikawa moves taking G_1 to G_2 .

FIGURE 25. Checking the *Ro2* move in Lemma 4.3c.

Note also that if we are presented with marked graph diagrams of G_1 and G_2 , then the marked graph diagrams are related by a sequence of *Type I* and *Type II* Yoshikawa moves, by the above and the results of Kauffman [Kau] for diagrams of 4-regular rigid vertex spatial graphs. \square

5. AN APPLICATION TO BRUNNIAN 2-LINKS

In Yajima [Yaj] it is shown that a 2-link with one component is ribbon if and only if there is a 2-link in its isotopy class with a generic projection in which all singularities are double-point curves. This is re-proved and extended to 2-links with any number of components in Kanenobu and Shima [KS, Theorem 5.2] by an elementary local argument. Thus we will forego any further discussion of the true definition of ribbon 2-links, though the reader may find more information in [CKS, KS, Yaj].

A 2-link $L \subset \mathbb{R}^4$ with components $L = L_1 \cup \dots \cup L_r$ is **trivial** if it is isotopic to a 2-link contained in \mathbb{R}^3 and **Brunnian** if each 2-link $L \setminus L_i$ for $1 \leq i \leq r$ is trivial. An example of a Brunnian 2-link can be found in Yanagawa [Yan, Figure 1]. The link components are the k_i 's and one can see that the link has a projection with only double-point curve singularities, and hence is ribbon.

Before we prove Theorem 5.2, we describe an analog for classical links in \mathbb{R}^3 due to Lee and Jin [LJ]. If $L = X \cup Y \subset \mathbb{R}^3$ is a link where X and Y are trivial links, then there is a projection of L to \mathbb{R}^2 in which each link X and Y is projected to a disjoint union of simple closed curves,

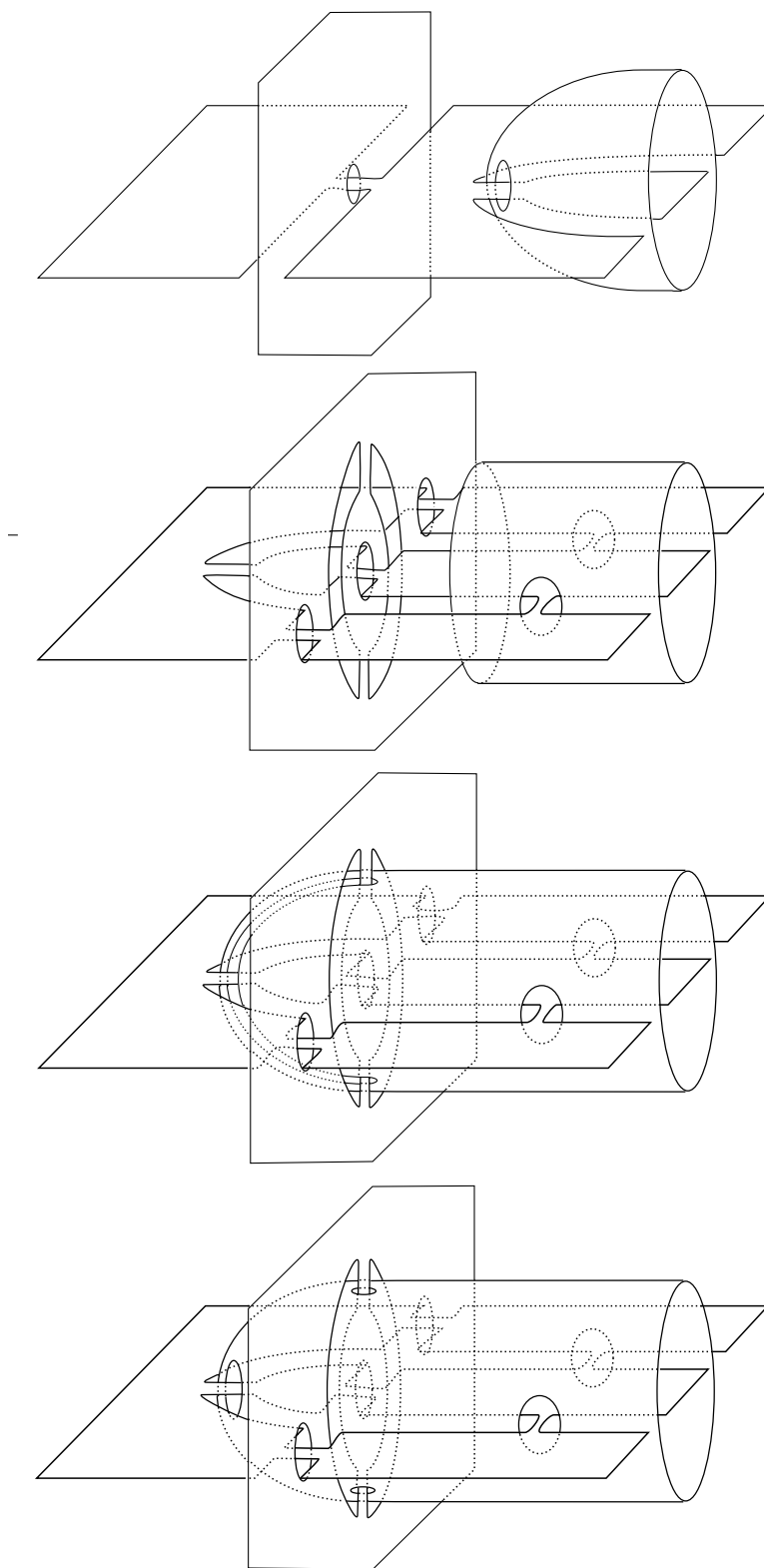
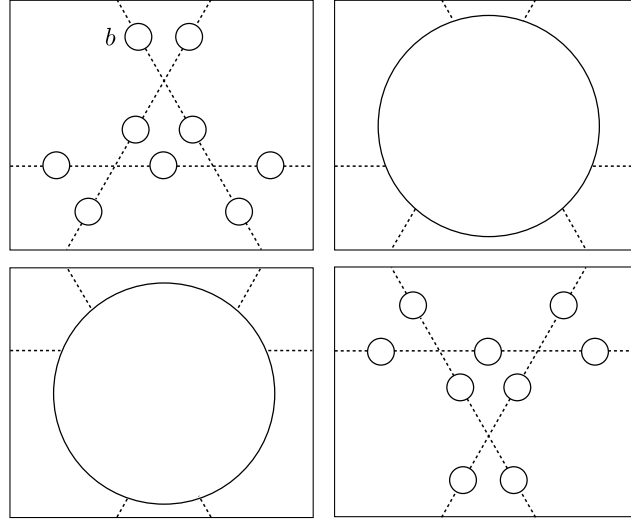


FIGURE 26. Checking the $Ro5^*$ move in Lemma 4.3c.

FIGURE 27. Checking the *Ro7* move in Lemma 4.3c.

and so all crossings in such a projection involve a strand of X and a strand Y . Lee and Jin prove the preceding statement by showing that given a projection of $X \cup Y$, arbitrary Reidemeister moves can be performed on the projection of X without changing the underlying projection of Y and vice versa. It follows that one can use Reidemeister moves to transform the projection of X into a disjoint union of simple curves without affecting the projection of Y , and then similarly the projection of Y can be transformed into a disjoint union of simple curves without affecting the already simplified projection of X . We will apply a similar philosophy to show that if $L = X \cup Y \subset \mathbb{R}^4$ is a 2-link and X and Y are trivial 2-links, then L has a broken surface diagram in which X and Y are both disjoint unions of embedded spheres. It will follow that L is ribbon.

Lemma 5.1. *a) If $R_1, R_2 \in \mathcal{S}_0$ are such that $\text{cap}(R_1) = \text{cap}(R_2)$ then R_1 and R_2 are related by a sequence of ab-moves.*

b) Let R_1, \dots, R_n be a sequence of ab-surfaces with R_i related to R_{i+1} by an ab-move. There exists another sequence S_1, \dots, S_n of ab-surfaces and an $m \leq n$ such that:

- $S_1 = R_1, S_n = R_n$,
- if $i < m$ then S_i is related to S_{i+1} by an ab-move that is not an $\overleftarrow{\text{ab1}}$ move,
- if $i \geq m$ then S_i is related to S_{i+1} by an $\overleftarrow{\text{ab1}}$ move.

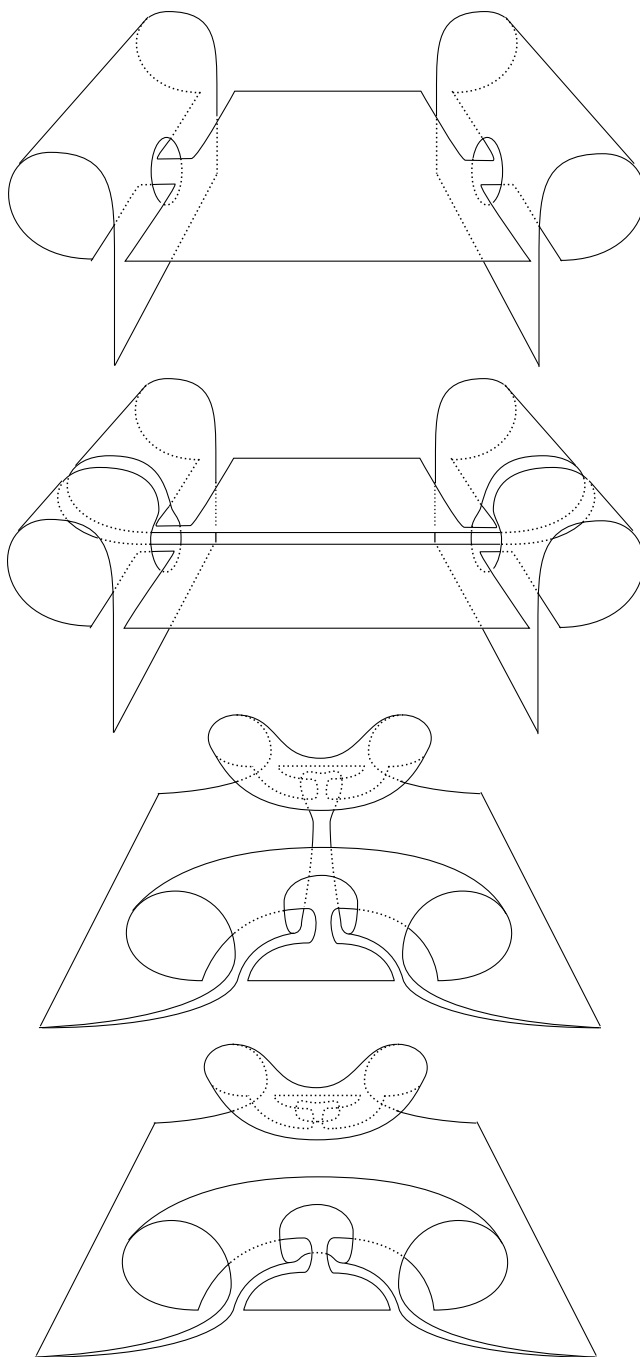


FIGURE 28. Checking the $Br1$ move in Lemma 4.3c.

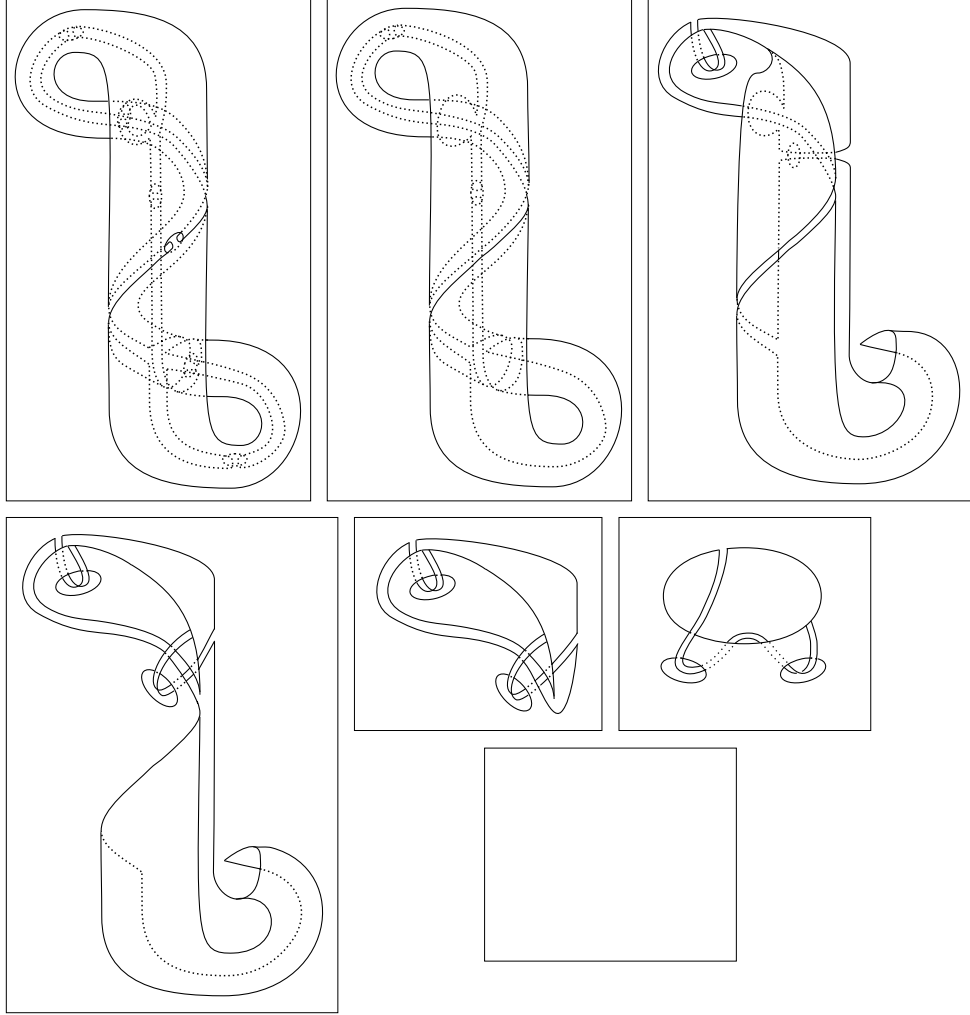


FIGURE 29. Checking a move equivalent to the $Br2$ move in Lemma 4.3c.

- c) Let $R_1 = P_1 \cup Q_1$ be an ab -surface where P_1, Q_1 are disjoint ab -surfaces. Assume an ab -move that is not an $\overleftarrow{ab1}$ move, takes P_1 to an ab -surface P_2 . Then the same ab -move can be performed on R_1 , taking R_1 to an ab -surface $R_2 = P'_2 \cup Q'_1$ where P'_2 is equivalent to P_2 and Q_2 is equivalent to Q_1 .

Proof. a) This is a consequence of the proof of Theorem 4.4.

- b) If we encounter an $\overleftarrow{ab1}$ move during the sequence R_1, \dots, R_n , we simply skip it until all other non- $\overleftarrow{ab1}$ moves have been performed. Skipping presents no problems for subsequent ab -moves.

- c) This is clear for $\overrightarrow{ab1}$ moves and $ab3$ moves. Each of these moves can be performed on P_1 without affecting Q_1 . If an $ab2$ move takes P_1 to P_2 then since Q_1 deformation retracts onto a 1-dimensional subgraph, it is clear we can push Q_1 away to not interfere with the $ab2$ move taking P_1 to P_2 . This pushing is not necessarily unique, so the union $R_2 = P'_2 \cup Q'_1$ is not necessarily uniquely determined up to equivalence, but P'_2 and Q'_1 are obviously uniquely determined up to equivalence.

□

Theorem 5.2. *If $L = X \cup Y \subset \mathbb{R}^4$ is a 2-link such that X and Y are disjoint trivial 2-links then L is ribbon. In particular, Brunnian 2-links are ribbon.*

Proof. Let $R = P \cup Q$ be an ab -surface representing L where $P = P_1 \cup \dots \cup P_r$ and $Q = Q_1 \cup \dots \cup Q_s$ are ab -surfaces representing X and Y respectively.

Since P represents a trivial 2-link, by Lemma 5.1a there is a sequence of ab -moves taking P to an ab -surface consisting of r disjoint annuli in \mathbb{R}^2 . By Lemma 5.1b we may assume that in this sequence, all $\overleftarrow{ab1}$ moves appear at the end. By Lemma 5.1c, we may perform this sequence of ab -moves on the surface R , and we choose to stop before the first $\overleftarrow{ab1}$ move. Thus we obtain a sequence of ab -moves taking R to an ab -surface $P_0 \cup Q'$ where Q' is equivalent to Q and P_0 is an ab -surface that is related by a sequence of $\overleftarrow{ab1}$ moves to a collection of annuli in \mathbb{R}^2 . Hence P_0 is itself equivalent to a surface contained in \mathbb{R}^2 . Similarly there is a sequence of ab -moves not including $\overleftarrow{ab1}$ moves, taking the ab -surface $P_0 \cup Q'$ to an ab -surface $P_0 \cup Q_0$ where now the ab -surface Q_0 is equivalent to an ab -surface contained in \mathbb{R}^2 .

Thus R is related by ab -moves to the ab -surface $P_0 \cup Q_0$. When we apply the cap function to $P_0 \cup Q_0$, we may choose a system of embedded disks $\{D_a^i\}_i$ and $\{D_b^j\}_j$ for P_0 so that $P_0 \cup \bigcup_i D_a^i \cup \bigcup_j D_b^j$ is a disjoint union of embedded spheres in \mathbb{R}^3 , and a system of embedded disks $\{E_a^k\}_k$ and $\{E_b^l\}_l$ for Q_0 so that $Q_0 \cup \bigcup_k E_a^k \cup \bigcup_l E_b^l$ is a disjoint union of embedded spheres in \mathbb{R}^3 as well. Thus after a perturbation

$$P_0 \cup \bigcup_i D_a^i \cup \bigcup_j D_b^j \cup Q_0 \cup \bigcup_k E_a^k \cup \bigcup_l E_b^l,$$

is a generic projection of L with only double-point curve singularities, and hence L is ribbon. Since Brunnian 2-links satisfy the original hypothesis, they too are ribbon. □

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E-mail address: oleg.chterental@mail.utoronto.ca